Topological spaces

Def: If X is a set, a topology on X is a collection $\mathcal{T}$ of subsets of X (where $U \subseteq X$ is open $\iff U \in \mathcal{T}$) satisfying the following properties.

1.) $\emptyset, X \in \mathcal{T}$

2.) If $\mathcal{T}' \subseteq \mathcal{T}$, then $\bigcup \mathcal{T}' \in \mathcal{T}$ (i.e. the union of open sets is open.)

3.) If $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$ (inductively, finite intersections of open sets are open).

A set equipped with a topology is called a topological space.

If X is a topological space w/ topology $\mathcal{T}$, then $U \subseteq X$ is open if $U \in \mathcal{T}$.

Just as with metric spaces, $F \subseteq X$ is closed if $X \setminus F$ is open.

Easy to check:

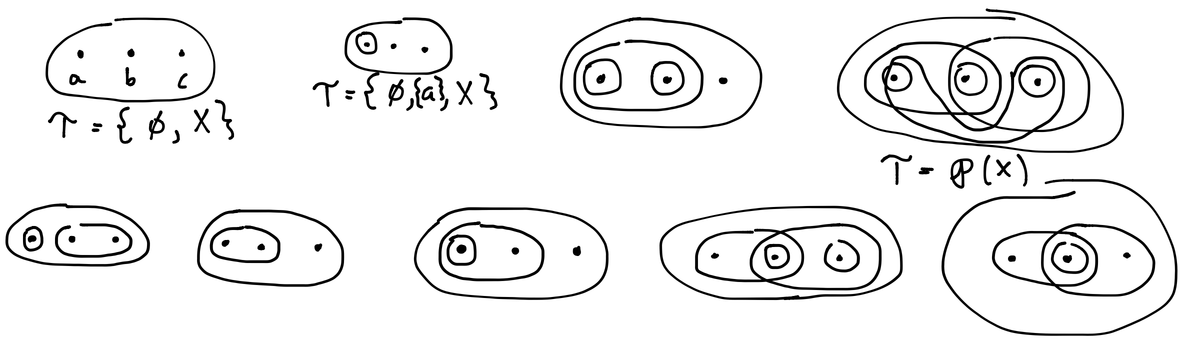
1.) $\emptyset, X$ are closed

2.) Arbitrary intersections of closed sets are closed.

3.) Finite unions of closed sets are closed.

Ex: The standard topology on $\mathbb{R}^n$, and more generally the topology induced by a metric space was described in the last section. We already proved that it satisfies all the axioms and thus is indeed a topological space.

Ex: Let $X = \{a, b, c\}$. What are the possible topologies on X?



You can get the rest by permuting the elements.

Ex: If X is a set, then the power set $\mathcal{P}(X)$ is a topology on X. It's called the <u>discrete topology</u>. Every set is open.

Ex: Let $\mathcal{T} = \{S \subseteq X \mid X - S \text{ is finite or } S = \emptyset\}$

i.e. the open sets are the subsets of X w/ finite complements.

Check that it's a topology:

1.) $\emptyset \in \mathcal{T}$ and $X - X = \emptyset$, so $X \in \mathcal{T}$

2.) If $S = \bigcup \mathcal{T}'$ then $X - S = \bigcap_{S' \in \mathcal{T}'} (X - S')$, which is finite (or X), since $X - S'$ is finite (or X).

3.) Similar to 2.)

This is called the finite complement topology or the cofinite topology.

Ex: Let X be an infinite set, $\mathcal{T} = \{S \subseteq X \mid S \text{ is finite or } S = X\}$

This is not a topology. For example, let $Y \subsetneq X$ be a proper infinite set. Then $Y = \bigcup_{y \in Y} \{y\}$, but $Y \notin \mathcal{T}$.